opinion, the view I put forward in 1877 that "new stars" are produced by the clash of meteor-swarms; and they have suggested some further tests of its validity.

We may hope since observations were made at Harvard and Potsdam very near the epoch of maximum brilliancy, that a subsequent complete discussion of the results obtained will very largely increase our knowledge. The interesting question arises whether we may not regard the changes in spectrum as indicating that the very violent intrusion of the denser swarm has been followed by its dissipation, and that its passage has produced movements in the sparser swarm which may eventuate in a subsequent condensation.

My best thanks are due to those I have named for assistance in this inquiry.

"Elastic Solids at Rest or in Motion in a Liquid." By C. CHREE, Sc.D., LL.D., F.R.S. Received November 19,—Read December 13, 1900.

§ 1. The problems dealt with in the present paper are probably of little practical importance; but they seem of considerable interest from the standpoint of dynamical theory. The hard and fast line which it is customary to draw between Rigid Dynamics and Elastic Solids has been discarded, and a more direct insight is thus obtained into the modes of transmission of force in solids.

Let us consider a solid of any homogeneous elastic material, possessed only of such symmetry of shape as will ensure that if it falls under gravity in a liquid, each element will move vertically. Take the origin of rectangular Cartesian co-ordinates at the centre of gravity, the axes of x and y being horizontal, and the axis of z being drawn vertically downwards. At time t let ζ be the depth of the C.G. below a horizontal plane in the liquid, the pressure on which is uniform and equal to Π. The existence of gaseous pressure on the liquid surface would only contribute to Π without modifying the general conditions of the problem.

Consider first the elementary hydrostatical theory, according to which the liquid pressure at any point x, y, z on the surface of the solid acts along the normal, and is equal to

$$\Pi + g\rho'(\zeta + z),$$

where ρ' is the density of the liquid, supposed uniform.

If the solid fall or rise very slowly, and the viscosity of the liquid is very small, the results based on the hydrostatical theory ought to give a close approximation to the truth.

If α, β, γ represent the elastic displacements, $\widehat{xx}$, $\widehat{xy}$, &c., the stresses in the notation of Todhunter and Pearson's 'History of Elasticity,' the body stress equations are of the type

$$\left.\begin{aligned} &\frac{d\widehat{xx}}{dx}+\frac{d\widehat{xy}}{dy}+\frac{d\widehat{xz}}{dz}-\rho\frac{d^2\alpha}{dt^2}=0\,,\\ &\frac{d\widehat{xy}}{dx}+\frac{d\widehat{yy}}{dy}+\frac{d\widehat{yz}}{dz}-\rho\frac{d^2\beta}{dt^2}=0\,,\\ &\frac{d\widehat{xz}}{dx}+\frac{d\widehat{yz}}{dy}+\frac{d\widehat{zz}}{dz}+\rho\left\{g-\frac{d^2(\zeta+\gamma)}{dt^2}\right\}=0 \end{aligned}\right\}\ldots\ldots\ldots\ldots \quad (1);$$

where ρ represents the density of the solid, g the acceleration of gravity.

The equations treat x, y, z as constants for each element of the solid, and so assume that the motion, if motion takes place, is purely translational.

If λ, μ, ν be the direction cosines of the outwardly directed normal at a point x, y, z, the surface equations are

$$\begin{aligned}(\lambda\widehat{xx}+\mu\widehat{xy}+\nu\widehat{xz})/\lambda &= (\lambda\widehat{xy}+\mu\widehat{yy}+\nu\widehat{yz})/\mu = (\lambda\widehat{xz}+\mu\widehat{yz}+\nu\widehat{zz})/\nu \\ &= -\Pi - g\rho'(\zeta+z)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).\end{aligned}$$

The equations (2) are satisfied by the assumption

$$\left.\begin{aligned}&\widehat{xx}=\widehat{yy}=\widehat{zz}=-\Pi-g\rho'(\zeta+z),\\ &\widehat{xy}=\widehat{xz}=\widehat{yz}=0\end{aligned}\right\}\ldots\ldots\ldots\ldots \quad (3).$$

Also the values (3) satisfy the body stress equations (1), provided

$$\begin{aligned}&\frac{d^2\alpha}{dt^2}=\frac{d^2\beta}{dt^2}=0\,,\\ &\rho\left\{\frac{d^2(\zeta+\gamma)}{dt^2}-g\right\}=-g\rho'\ \ldots\ldots\ldots\ldots\ldots\ldots \quad (4).\end{aligned}$$

We can satisfy (4) by assuming

$$\begin{aligned}&\frac{d^2\gamma}{dt^2}=0\,,\\ &\zeta=\text{const.}+\tfrac{1}{2}g\frac{\rho-\rho'}{\rho}t^2\ \ldots\ldots\ldots\ldots\ldots\ldots \quad (5).\end{aligned}$$

For brevity, the constant in (5) will be supposed to be zero.

The result (5) is of course that given by ordinary elementary methods for the accelerated motion of a solid rising or falling in a liquid of different density.

On looking more closely into the matter an inconsistency manifests itself. Supposing for mathematical simplicity that the solid is isotropic, of bulk modulus k, we find that the displacements answering to (3) are given by

$$\left.\begin{aligned} \alpha/x = \beta/y &= -\{\Pi + g\rho'(\zeta + z)\}/3k, \\ \gamma &= -[\Pi z + g\rho'\{z(\zeta+z) - \tfrac{1}{2}(x^2+y^2+z^2)\}]/3k \end{aligned}\right\} \quad \ldots\ldots \quad (6).$$

The inconsistency consists in the fact that, by (6), α, β, γ contain terms in ζ, and so by (5) terms in t^2, while above it was assumed that $d^2\alpha/dt^2$, &c., vanished. It thus appears that the solution embodied in (3) and (6) is valid and complete only when ζ does not vary as t^2, *i.e.*, only when the solid is at rest or moving with uniform velocity in the liquid.

Though thus restricted, the solution is notable from its simplicity and generality, as applicable to any homogeneous solid (free from cavities) at rest in a liquid of equal density.

The values (3) for the stresses apply irrespective of the species of elasticity. The displacements are given by (6) only when the material is isotropic, but corresponding expressions are immediately obtainable for materials of greater complexity. If for instance we have material symmetrical with respect to the co-ordinate planes, we have

$$\left.\begin{aligned} \alpha &= -x\{\Pi + g\rho'(\zeta+z)\}(1-\eta_{12}-\eta_{13})/E_1, \\ \beta &= -y\{\Pi + g\rho'(\zeta+z)\}(1-\eta_{21}-\eta_{23})/E_2, \\ \gamma &= -z\{\Pi + g\rho'(\zeta+\tfrac{1}{2}z)\}(1-\eta_{31}-\eta_{32})/E_3 \\ &\quad + \tfrac{1}{2}g\rho'\left\{\frac{x^2}{E_1}(1-\eta_{12}-\eta_{13}) + \frac{y^2}{E_2}(1-\eta_{21}-\eta_{23})\right\} \end{aligned}\right\} \quad \ldots \quad (7).$$

Here E_1, E_2, E_3 are the three principal Young's moduli, while η_{12}, η_{13}, &c., are the corresponding Poisson's ratios.

§ 2. Presently we shall consider the equilibrium problem in greater detail. Meanwhile, in the case of uniformly accelerated motion, we shall obtain a self-consistent solution for a sphere, or any form of solid ellipsoid, under the conditions assumed in § 1.

The procedure to be adopted is the same for all species of elastic material. If for definiteness we suppose the material symmetrical with respect to the three co-ordinate planes, we first assume that the stresses (3) and displacements (7) form part—but only part—of the complete solution, ζ being given by (5). Then substituting from (7) in the body stress equations (1), we find that the stresses of the *supplementary* solution, as we may call it, must satisfy

$$\left.\begin{aligned} \frac{d\widehat{xx}}{dx}+\frac{d\widehat{xy}}{dy}+\frac{d\widehat{xz}}{dz} &= -\mathrm{P}\rho x\,, \\ \frac{d\widehat{xy}}{dx}+\frac{d\widehat{yy}}{dy}+\frac{d\widehat{yz}}{dz} &= -\mathrm{Q}\rho y\,, \\ \frac{d\widehat{xz}}{dx}+\frac{d\widehat{yz}}{dy}+\frac{d\widehat{zz}}{dz} &= -\mathrm{R}\rho z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8);$$

where

$$\left.\begin{aligned} \mathrm{P}\rho &\equiv g^2\rho'(\rho-\rho')(1-\eta_{12}-\eta_{13})/\mathrm{E}_1\,, \\ \mathrm{Q}\rho &\equiv g^2\rho'(\rho-\rho')(1-\eta_{21}-\eta_{23})/\mathrm{E}_2\,, \\ \mathrm{R}\rho &\equiv g^2\rho'(\rho-\rho')(1-\eta_{31}-\eta_{32})/\mathrm{E}_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (9).$$

The surface equations to be satisfied by the supplementary solution are

$$\lambda\widehat{xx}+\mu\widehat{xy}+\nu\widehat{xz} = \lambda\widehat{xy}+\mu\widehat{yy}+\nu\widehat{yz} = \lambda\widehat{xz}+\mu\widehat{yz}+\nu\widehat{zz} = 0 \ldots \quad (10).$$

The problem thus resolves itself into that of an ellipsoid acted on solely by bodily forces derivable from the potential

$$\tfrac{1}{2}(\mathrm{P}x^2+\mathrm{Q}y^2+\mathrm{R}z^2)\,.$$

This problem was solved by me in 1894 for isotropic* materials, and in 1899 I extended the solution to æolotropic† ellipsoids. We can thus derive a satisfactory supplementary solution from the sources specified. Finally adding the stresses of the supplementary solution to the stresses (3), and the displacements to the displacements (7), we have a consistent and complete solution of the problem presented by a heavy ellipsoid in a homogeneous liquid, when the action of the liquid is supposed that given by elementary hydrostatics.

§ 3. The supplementary solution, though simple in type, contains terms which are of great length when the ellipsoid has three unequal axes, and is of a complex kind of æolotropy. It will thus perhaps suffice to select for illustration the simple case of an isotropic sphere of radius a.

Denoting Young's modulus by E, Poisson's ratio by η, and writing r^2 for $x^2 + y^2 + z^2$, we have in full

* 'Roy. Soc. Proc.,' vol. 58, p. 39; 'Quarterly Journal of Pure and Applied Mathematics,' vol. 27, p. 338.

† 'Camb. Phil. Soc. Trans.,' vol. 17, p. 201.

$$\left.\begin{aligned}
\widehat{xx} &= -\Pi - g\rho'\left(z+\tfrac{1}{2}g\frac{\rho-\rho'}{\rho}t^2\right)\\
&\quad+\frac{g^2\rho'(\rho-\rho')(1-2\eta)}{10E(1-\eta)}\{(3-\eta)(a^2-x^2)-(1+3\eta)(y^2+z^2)\},\\
\widehat{yy} &= -\Pi - g\rho'\left(z+\tfrac{1}{2}g\frac{\rho-\rho'}{\rho}t^2\right)\\
&\quad+\frac{g^2\rho'(\rho-\rho')(1-2\eta)}{10E(1-\eta)}\{(3-\eta)(a^2-y^2)-(1+3\eta)(x^2+z^2)\},\\
\widehat{zz} &= -\Pi - g\rho'\left(z+\tfrac{1}{2}g\frac{\rho-\rho'}{\rho}t^2\right)\\
&\quad+\frac{g^2\rho'(\rho-\rho')(1-2\eta)}{10E(1-\eta)}\{(3-\eta)(a^2-z^2)-(1+3\eta)(x^2+y^2)\},\\
\widehat{xy}/xy &= \widehat{xz}/xz = \widehat{yz}/yz = -g^2\rho'(\rho-\rho')(1-2\eta)^2\div\{5E(1-\eta)\}
\end{aligned}\right\}\ \ldots (11);$$

$$\left.\begin{aligned}
\alpha/x = \beta/y &= -\frac{1-2\eta}{E}\left\{\Pi+g\rho'\left(z+\tfrac{1}{2}g\frac{\rho-\rho'}{\rho}t^2\right)\right\}\\
&\quad+\frac{g^2\rho'(\rho-\rho')(1-2\eta)^2}{10E^2(1-\eta)}\{(3-\eta)a^2-(1+\eta)r^2\},\\
\gamma &= -\frac{1-2\eta}{E}\left\{\Pi z+g\rho'z\left(z+\tfrac{1}{2}g\frac{\rho-\rho'}{\rho}t^2\right)-\tfrac{1}{2}g\rho'r^2\right\}\\
&\quad+\frac{g^2\rho'(\rho-\rho')(1-2\eta)^2 z}{10E^2(1-\eta)}\{(3-\eta)a^2-(1+\eta)r^2\}
\end{aligned}\right\}\ \ldots (12).$$

The terms in g^2 constitute what has been called above the supplementary solution. In the case alike of the stresses and of the displacements they are exactly the same as if the sphere were under a self-gravitative force which followed the ordinary gravitational law, and which had for its accelerative value at the surface of the sphere

$$\frac{g^2\rho'(\rho'-\rho)}{\rho}\,\frac{1-2\eta}{E}\,a\,.$$

This imaginary gravitative action represents attraction or repulsion between elements of the solid according as $\rho-\rho'$ is negative or positive. It is thus an attraction when the sphere rises in a heavier liquid, a repulsion when it sinks in a lighter. The smaller $1-2\eta$, or in general the less compressible the solid, the smaller is the effect of this imaginary gravitative force relative to that of the hydrostatic pressure $\Pi + g\rho'(z+\zeta)$; on the other hand its relative importance increases rapidly with the size of the sphere.

Representing by dashed letters the parts of the displacements depending on $\rho-\rho'$, we have

$$\alpha'/x = \beta'/y = \gamma'/z$$
$$= \frac{g^2\rho'(\rho-\rho')}{2\rho E}(1-\eta)\left[\frac{\rho(1-2\eta)}{5E}\{(3-\eta)a^2-(1+\eta)r^2\}-t^2\right] \ldots (13).$$

At the very beginning of the motion, the expression inside the square bracket is positive for all values of r; but as t increases it changes sign, first at the surface, last close to the centre of the sphere. If ζ_a, ζ_0 represent the distances fallen when the expression vanishes at the surface and at the centre respectively, we have

$$\left.\begin{aligned}\zeta_a/a &= (1-\eta)g(\rho-\rho')a/15k,\\ \zeta_0/a &= (3-\eta)g(\rho-\rho')a/30k\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (14).$$

Unless a is enormously large, ζ_a and ζ_0 must be extremely small for any ordinary elastic material.

In reality, in order to be instantaneously at rest, the sphere would require to be supported or acted on by some suddenly suppressed force, or to be in the act of reversing some previously impressed motion. The elastic strains and stresses might initially retain the impress of the pre-existing state of matters, and there are thus special sources of uncertainty affecting the applicability of (14) to actual conditions, which should not be lost sight of.

§ 4. The problem just considered has been advanced as showing how under a consistent dynamical system, producing uniform acceleration in a straight line, there appear elastic strains and stresses which simulate the action of self-gravitation in the material in motion. The conditions postulated do not answer exactly to what happens when a real solid moves through real liquid at the earth's surface. Under such circumstances the action between solid and liquid is not fully represented by the hydrostatic pressure. If the fluid be "perfect," ordinary hydrodynamical theory* gives for the pressure p on the surface of the sphere, supposing u the velocity,

$$p = \Pi + g\rho'(\zeta+z) + \rho'(\tfrac{1}{2}a\dot{u}P_1 + \tfrac{3}{4}u^2P_2 - \tfrac{1}{4}u^2) \ldots\ldots\ldots\ldots (15),$$

where P_1, P_2 are zonal harmonics, whose axis is the vertical diameter. We shall now consider this case, on the hypothesis that the velocity is so small that terms in u^2 are negligible. Instead of (3) and (6) we find for the stresses and displacements, the material being supposed isotropic,

$$\left.\begin{aligned}\widehat{xx} = \widehat{yy} = \widehat{zz} &= -\Pi - g\rho'(\zeta+z) - \tfrac{1}{2}\dot{u}\rho' z,\\ \widehat{xy} = \widehat{xz} = \widehat{yz} &= 0\end{aligned}\right\} \quad \ldots\ldots\ldots (16);$$

* *Cf.* Lamb's 'Hydrodynamics,' Art. 91.

$$\left.\begin{aligned}\alpha/x = \beta/y &= -\frac{1-2\eta}{\mathrm{E}}\{\Pi + g\rho'(\zeta+z) + \tfrac{1}{2}\dot{u}\rho' z\},\\ \gamma &= -\frac{1-2\eta}{\mathrm{E}}[\Pi z + g\rho' z\zeta + \tfrac{1}{2}\rho'(g+\tfrac{1}{2}\dot{u})(z^2 - x^2 - y^2)]\end{aligned}\right\}\ldots (17).$$

Instead of (4) we have

$$\rho\left\{\frac{d^2(\zeta+\gamma)}{dt^2} - g\right\} = -g\rho' - \tfrac{1}{2}\dot{u}\rho'.$$

Also
$$\dot{u} \equiv d^2\zeta/dt^2,$$

thus, if $d^2\gamma/dt^2$ be omitted, we have

$$(\rho + \tfrac{1}{2}\rho')\frac{d^2\zeta}{dt^2} = g(\rho - \rho'),$$

or
$$\zeta = \text{constant} + \tfrac{1}{2}g\frac{\rho-\rho'}{\rho+\tfrac{1}{2}\rho'}t^2 \ldots\ldots\ldots\ldots\ldots\ldots (18).$$

This is, of course, only the well-known result, that the dynamical action of the liquid may be regarded as adding to the mass of the sphere that of a hemisphere of the liquid.* We may suppose the constant in (18) to be zero, suitably interpreting Π.

As in the first case considered, the existence of t^2 in ζ and, consequently, in α, β, γ, makes a supplementary solution necessary. The stresses of the supplementary solution must satisfy the surface equations (10) as well as the following body stress equations:

$$\left(\frac{d\widehat{xx}}{dx} + \frac{d\widehat{xy}}{dy} + \frac{d\widehat{xz}}{dz}\right)\Big/ x = \left(\frac{d\widehat{xy}}{dx} + \frac{d\widehat{yy}}{dy} + \frac{d\widehat{yz}}{dz}\right)\Big/ y$$
$$= \left(\frac{d\widehat{xz}}{dx} + \frac{d\widehat{yz}}{dy} + \frac{d\widehat{zz}}{dz}\right)\Big/ z = -\frac{1-2\eta}{\mathrm{E}}\,\frac{2g^2\rho\rho'(\rho-\rho')}{2\rho+\rho'} \ldots\ldots\ldots (19).$$

It will be observed that the retention of the term in $\dot{u}$ in the pressure has only modified (reduced) the acceleration without altering the type of the supplementary solution. It will thus suffice to record the complete expressions for the displacements, viz.,

$$\left.\begin{aligned}\alpha/x = \beta/y = &-\frac{1-2\eta}{\mathrm{E}}\left[\Pi + \frac{3g\rho\rho' z}{2\rho+\rho'} + g^2\frac{\rho'(\rho-\rho')}{2\rho+\rho'}t^2\right]\\ &+\frac{(1-2\eta)^2}{5(1-\eta)\mathrm{E}^2}\,\frac{g^2\rho\rho'(\rho-\rho')}{2\rho+\rho'}[(3-\eta)a^2 - (1+\eta)r^2],\\ \gamma = &-\frac{1-2\eta}{\mathrm{E}}\left[\Pi z + \frac{3g\rho\rho'}{2(2\rho+\rho')}(z^2-x^2-y^2) + \frac{g^2\rho'(\rho-\rho')}{2\rho+\rho'}zt^2\right]\\ &+\frac{(1-2\eta)^2}{5(1-\eta)\mathrm{E}^2}\,\frac{g^2\rho\rho'(\rho-\rho')}{2\rho+\rho'}z[(3-\eta)a^2-(1+\eta)r^2]\end{aligned}\right\}(20).$$

* *Cf.* Lamb's 'Hydrodynamics,' Art. 91; or Basset's 'Treatise on Hydrodynamics,' Art. 182.

In obtaining this solution we have neglected terms in u^2, *i.e.*, terms in $(d\zeta/dt)^2$ or $g^2t^2(\rho-\rho')^2/(\rho+\frac{1}{2}\rho')^2$, in the expression (15), while there appear in the solution terms containing $g^2t^2(\rho-\rho')/(2\rho+\rho')$. Thus our work is consistent only when $(\rho-\rho')/\rho$ is small, and even when this is the case the fact that u^2 increases as t^2 involves a restriction which should not be overlooked. It would not, I think, be a very difficult matter to obtain a complete solution answering to the full value (15) of p. Treating u^2 at first as a constant, we could at once write down, from my general solution* for the isotropic elastic sphere, the displacements answering to the surface pressure $\frac{1}{4}\rho'u^2(3\mathrm{P}_2-1)$; but the explicit determination of the corresponding supplementary solution would be much more laborious than in the first case treated above.

§ 5. When ρ' and ρ are equal, and u^2 is thus really constant, the complete values of the stresses and displacements answering to the surface pressure (15) are as follows :—

$$\left.\begin{aligned}
\widehat{xx} &= -\Pi - g\rho'(z+ut) + \tfrac{1}{4}\rho'u^2 + \tfrac{3}{8}\frac{\rho'u^2a^{-2}}{7+5\eta}\,[(7+2\eta)a^2 \\
&\qquad + 3\eta(x^2+5y^2) - 3(7+6\eta)z^2], \\
\widehat{yy} &= -\Pi - g\rho'(z+ut) + \tfrac{1}{4}\rho'u^2 + \tfrac{3}{8}\frac{\rho'u^2a^{-2}}{7+5\eta}\,[(7+2\eta)a^2 \\
&\qquad + 3\eta(5x^2+y^2) - 3(7+6\eta)z^2], \\
\widehat{zz} &= -\Pi - g\rho'(z+ut) + \tfrac{1}{4}\rho'u^2 - \tfrac{3}{8}\frac{\rho'u^2a^{-2}}{7+5\eta}\,[2(7+2\eta)a^2 \\
&\qquad - 3(7+\eta)(x^2+y^2) + 6\eta z^2], \\
\widehat{xy} &= -9\rho'u^2\eta xya^{-2} \div [2(7+5\eta)], \\
\widehat{xz}/xz &= \widehat{yz}/yz = 9\rho'u^2\eta a^{-2} \div [4(7+5\eta)]
\end{aligned}\right\} \ldots (21);$$

$$\left.\begin{aligned}
\alpha/x = \beta/y &= -\frac{1-2\eta}{\mathrm{E}}\,[\Pi - \tfrac{1}{4}\rho'u^2 + g\rho'(z+ut)] \\
&\quad + \tfrac{3}{8}\frac{\rho'u^2(1+\eta)a^{-2}}{\mathrm{E}(7+5\eta)}[(7+2\eta)a^2 - 6\eta(x^2+y^2) - 3(7-8\eta)z^2], \\
\gamma &= -\frac{1-2\eta}{\mathrm{E}}\,[(\Pi - \tfrac{1}{4}\rho'u^2)z + g\rho'\{utz + \tfrac{1}{2}(z^2-x^2-y^2)\}] \\
&\quad - \tfrac{3}{8}\frac{\rho'u^2(1+\eta)za^{-2}}{\mathrm{E}(7+5\eta)}\,[2(7+2\eta)a^2 - 3(7-6\eta)(x^2+y^2) - 12\eta z^2]
\end{aligned}\right\} (22).$$

§ 6. In real liquids viscosity is more or less present, and as the hydrodynamical equations have been solved for the case of an ellipsoid

* 'Camb. Phil. Soc. Trans.,' vol. 14, p. 250.

when the retarding action of viscosity neutralises the acceleration due to gravity, it is worth considering. The hydrodynamical solution really assumes the velocity to be small, and the ellipsoid to be so remote from the surface and other boundaries as to be practically in an infinite liquid.

It is not very difficult to deduce from the formulæ in Lamb's 'Hydrodynamics,' Art. 296,—though I have not seen the result noticed —that the viscous surface action reduces to a force $f\varpi$ per unit surface, opposite to the direction of motion, ϖ being the perpendicular from the centre on the tangent plane, and f a constant. The recognition of this fact saves us from the labour of considering the general expressions for the hydrodynamical pressures, which are of a very complicated nature.

As the motion is steady, the body stress equations are

$$\frac{d\widehat{xx}}{dx}+\frac{d\widehat{xy}}{dy}+\frac{d\widehat{xz}}{dz}=\frac{d\widehat{xy}}{dx}+\frac{d\widehat{yy}}{dy}+\frac{d\widehat{yz}}{dz}=\frac{d\widehat{xz}}{dx}+\frac{d\widehat{yz}}{dy}+\frac{d\widehat{zz}}{dz}+g\rho = 0 \ldots (23);$$

while the surface equations are—a, b, c being the semi-axes of the ellipsoid—

$$\left.\begin{aligned} a^{-2}x\widehat{xx}+b^{-2}y\widehat{xy}+c^{-2}z\widehat{xz} &= -a^{-2}x\{\Pi+g\rho'(\zeta+z)\}, \\ a^{-2}x\widehat{xy}+b^{-2}y\widehat{yy}+c^{-2}z\widehat{yz} &= -b^{-2}y\{\Pi+g\rho'(\zeta+z)\}, \\ a^{-2}x\widehat{xz}+b^{-2}y\widehat{yz}+c^{-2}z\widehat{zz} &= -c^{-2}z\{\Pi+g\rho'(\zeta+z)\}-f \end{aligned}\right\} (24).$$

The surface equations are satisfied by

$$\left.\begin{aligned} \widehat{xx} &= -\Pi-g\rho'(\zeta+z)+(a^2/c^2)fz, \\ \widehat{yy} &= -\Pi-g\rho'(\zeta+z)+(b^2/c^2)fz, \\ \widehat{zz} &= -\Pi-g\rho'(\zeta+z)-fz, \\ \widehat{xy} &= 0, \\ \widehat{xz}/x &= \widehat{yz}/y = -f \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (25).$$

The values (25) also satisfy the body stress equations (23), provided

$$-3f+g(\rho-\rho') = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26).$$

As $$\iint f\varpi dS = 3f\,.\,\tfrac{4}{3}\pi abc,$$

when the integral is taken over the surface of the ellipsoid, (26) is simply equivalent to the condition that the motion is not accelerated, or that

$$\zeta = ut.$$

where u is a constant. As to the value of u, it has been proved that the total viscous resistance to the motion is*

$$16\pi\mu' uabc/(\chi_0 + c^2\gamma_0),$$

where μ' is the viscosity, and

$$\chi_0 \equiv abc\int_0^\infty [(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)]^{-\frac{1}{2}}d\lambda,$$

$$\gamma_0 \equiv abc\int_0^\infty [(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)^3]^{-\frac{1}{2}}d\lambda.$$

But this resistance is also equal to $g(\rho - \rho')\frac{4}{3}\pi abc$ [or to $\iint f\varpi dS$], thus

$$u = g(\rho-\rho')(\chi_0 + c^2\gamma_0)/12\mu'.$$

Substituting for ζ and f in (25), we have

$$\left.\begin{aligned}
\widehat{xx} &= -\Pi - g\rho'(z+ut) + \tfrac{1}{3}g(\rho-\rho')a^2z/c^2, \\
\widehat{yy} &= -\Pi - g\rho'(z+ut) + \tfrac{1}{3}g(\rho-\rho')b^2z/c^2, \\
\widehat{zz} &= -\Pi - g\rho'(z+ut) - \tfrac{1}{3}g(\rho-\rho')z, \\
\widehat{xy} &= 0, \\
\widehat{xz}/x &= \widehat{yz}/y = -\tfrac{1}{3}g(\rho-\rho')
\end{aligned}\right\} \quad\text{.............}(27).$$

The corresponding displacements, supposing the material isotropic, are

$$\left.\begin{aligned}
\alpha &= -\frac{1-2\eta}{E}x[\Pi + g\rho'(z+ut)] + \frac{g(\rho-\rho')}{3E}xz\left[\frac{a^2}{c^2} + \eta\left(1-\frac{b^2}{c^2}\right)\right], \\
\beta &= -\frac{1-2\eta}{E}y[\Pi + g\rho'(z+ut)] + \frac{g(\rho-\rho')}{3E}yz\left[\frac{b^2}{c^2} + \eta\left(1-\frac{a^2}{c^2}\right)\right], \\
\gamma &= -\frac{1-2\eta}{E}[\Pi z + g\rho' utz + \tfrac{1}{2}g\rho'(z^2-x^2-y^2)] \\
&\qquad - \frac{g(\rho-\rho')z^2}{6E}\left(1+\eta\frac{a^2+b^2}{c^2}\right) \\
&\qquad - \frac{g(\rho-\rho')}{6E}\left[x^2\left(2+3\eta+\frac{a^2-\eta b^2}{c^2}\right) + y^2\left(2+3\eta+\frac{b^2-\eta a^2}{c^2}\right)\right]
\end{aligned}\right\} \quad (28).$$

§ 7. The terms inside the first brackets in (28) contain Π or $g\rho'$, and represent displacements which vary only with the depth of the element or its distance from the centre of the ellipsoid. The terms containing

* *Cf.* Lamb's 'Hydrodynamics,' Art. 296.

$g(\rho - \rho')$, on the other hand, depend largely on the shape of the ellipsoid.

Thus, denoting them by α', β', γ', we have approximately, in the case of a very elongated ellipsoid, whose long axis is vertical,

$$\left.\begin{aligned} \alpha'/x = \beta'/y &= g(\rho-\rho')\eta z/3\mathrm{E}, \\ \gamma' &= -g(\rho-\rho')\,[z^2+(2+3\eta)\,(x^2+y^2)]/6\mathrm{E} \end{aligned}\right\} \quad \ldots\ldots(29);$$

and, except in the immediate vicinity of the central section $z = 0$, we may take in place of (29)

$$\alpha'/x\eta = \beta'/y\eta = -\gamma'/(\tfrac{1}{2}z) = g(\rho-\rho')z/3\mathrm{E} \ldots\ldots\ldots\ldots(30).$$

In a very flat ellipsoid, approximating to a disc, with the short axis vertical, we have approximately

$$\left.\begin{aligned} \alpha' &= g(\rho-\rho')xz(a^2-\eta b^2)/(3\mathrm{E}c^2), \\ \beta' &= g(\rho-\rho')yz(b^2-\eta a^2)/(3\mathrm{E}c^2), \\ \gamma' &= -g(\rho-\rho')\,[(a^2-\eta b^2)x^2+(b^2-\eta a^2)y^2+\eta(a^2+b^2)z^2]\div(6\mathrm{E}c^2) \end{aligned}\right\} (31).$$

Except close to the vertical diameter, the terms in z^2 in γ' would be relatively negligible, while, in general, α' and β' would be small compared to γ'.

In the case of the sphere it is perhaps more convenient to record the complete solution, viz.,

$$\left.\begin{aligned} \widehat{xx} = \widehat{yy} &= -\Pi - g\rho' ut + \tfrac{1}{3}g(\rho-4\rho')z, \\ \widehat{zz} &= -\Pi - g\rho' ut - \tfrac{1}{3}g(\rho+2\rho')z, \\ \widehat{xy} &= 0, \\ \widehat{xz}/x = \widehat{yz}/y &= -\tfrac{1}{3}g(\rho-\rho') \end{aligned}\right\} \quad \ldots\ldots(32);$$

$$\left.\begin{aligned} \alpha/x = \beta/y &= -\frac{1-2\eta}{\mathrm{E}}[\Pi+g\rho'(z+ut)]+\tfrac{1}{3}g(\rho-\rho')z/\mathrm{E}, \\ \gamma &= -\frac{1-2\eta}{\mathrm{E}}[(\Pi+g\rho' ut)z+\tfrac{1}{2}g\rho'(z^2-x^2-y^2)] \\ &\quad -\frac{g(\rho-\rho')}{6\mathrm{E}}[(3+2\eta)(x^2+y^2)+(1+2\eta)z^2] \end{aligned}\right\} \ldots(33).$$

[*March* 13, 1901.]—The paper as originally presented to the Society dealt briefly with two or three other details. It showed how the solution in § 6 depended not on the viscous resistance varying as the first power of the velocity in the final state, but on its varying over the

surface as the perpendicular on the tangent plane. In particular, if, in accordance with Mr. Allen's experiments,* there be possible forms of final uniform motion for a sphere in which the resistance varies as $u^{\frac{3}{2}}$ or u^2 (u being the velocity), it was shown that the solution would still be of the form of (32) and (33), provided the distribution of the viscous resistance happens to remain unchanged.

It was pointed out that in an isotropic solid, free of cavities, at rest in a liquid, the *stresses* are everywhere the same as if each element were separately subjected to the pressure answering to its depth; but that when cavities exist in the solid the state of matters is altered. As an example, a complete solution was given for a hollow spherical shell fully immersed.

It was shown that, in a completely solid body, the greatest strain and maximum stress-difference theories agreed in indicating no tendency to rupture, but that when cavities existed, it was otherwise; in particular, that in the spherical shell there is on either theory a tendency to rupture, greatest at the lowest point, which approximately in a thin shell varies directly as the depth and inversely as the thickness of the shell.

"On the Heat dissipated by a Platinum Surface at High Temperatures. Part IV.†—High-pressure Gases." By J. E. PETAVEL, A.M.I.C.E., A.M.I.E.E., John Harling Fellow of Owens College, Manchester. Communicated by Professor SCHUSTER, F.R.S. Received February 7,—Read March 7, 1901.

(Abstract.)

The rate of cooling of a hot body in gases at pressures up to one atmosphere has received considerable attention, but with regard to gases at high pressures practically no data were up to the present available. It was thought therefore that an experimental investigation of the subject might prove of some interest.

The experiments were carried out with a horizontal cylindrical radiator contained in a strong steel enclosure, the enclosure being maintained at about 18° C. by a water circulation.

It is shown that the rate at which heat is dissipated by the radiator may be expressed by the following formula—

$$E = ap^{\alpha} + bp^{\beta}\vartheta,$$

where E = emissivity in C.G.S. units = total amount of heat dissi-

* 'Phil. Mag.,' September and November, 1900.

† For Parts I, II and III see 'Phil., Trans.,' A, vol. 191, p. 501, 1898.